XIX. *On the Differential Equations which determine the form of the Roots of Algebraic Equations.*

By GEORGE BOOLE, *F.R.S., Professor of Mathematics in Queen's College, Cork.*

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1. Mr. HARLEY* has shown that any root of the equation

$$y^n-ny+(n-1)x=0$$

satisfies the differential equation

$$y-\frac{\left(\mathrm{D}-\frac{2n-1}{n}\right)\left(\mathrm{D}-\frac{3n-2}{n}\right)\ldots\left(\mathrm{D}-\frac{n^2-n+1}{n}\right)}{\mathrm{D}(\mathrm{D}-1)\ldots(\mathrm{D}-n+1)}e^{(n-1)\theta}y=0, \quad . \quad . \quad . \quad . \quad (1)$$

in which $e^\theta=x$, and $\mathrm{D}=\frac{d}{d\theta}$, provided that n be a positive integer greater than 2. This result, demonstrated for particular values of n, and raised by induction into a general theorem, was subsequently established rigorously by Mr. CAYLEY by means of LAGRANGE'S theorem.

For the case of $n=2$, the differential equation was found by Mr. HARLEY to be

$$y-\frac{\mathrm{D}-\frac{3}{2}}{\mathrm{D}}e^\theta y=\tfrac{1}{2}e^\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Solving these differential equations for the particular cases of $n=2$ and $n=3$, Mr. HARLEY arrived at the actual expression of the roots of the given algebraic equation for these cases. That all algebraic equations up to the fifth degree can be reduced to the above trinomial form, is well known.

A solution of (1) by means of definite triple integrals in the case of $n=4$ has been published by Mr. W. H. L. RUSSELL; and I am informed that a general solution of the equation by means of a definite single integral has been obtained by the same analyst.

While the subject seems to be more important with relation to differential than with reference to algebraic equations, the connexion into which the two subjects are brought must itself be considered as a very interesting fact. As respects the former of these subjects, it may be observed that it is a matter of quite fundamental importance to ascertain for what forms of the function ϕ (D), equations of the type

$$u+\phi(\mathrm{D})e^{n\theta}u=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

admit of finite solution. We possess theorems which enable us to deduce from each known integrable form an infinite number of others. Yet there is every reason to think

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that the number of really primary forms—of forms the knowledge of which, in combination with such known theorems, would enable us to solve all equations of the above type that are finitely solvable—is extremely small. It will, indeed, be a most remarkable conclusion, should it ultimately prove that the forms in question stand in absolute and exclusive connexion with the class of algebraic equations here considered.

The following paper is a contribution to the general theory under the aspect last mentioned. In endeavouring to solve Mr. HARLEY'S equation by definite integrals, I was led to perceive its relation to a more general equation, and to make this the subject of investigation. The results will be presented in the following order:—

First, I shall show that if u stand for the mth power of any root of the algebraic equation

$$y^n-xy^{n-1}-1=0,$$

then u, considered as a function of x, will satisfy the differential equation

$$[\mathrm{D}]^n u+\left[\frac{n-1}{n}\mathrm{D}+\frac{m}{n}-1\right]^{n-1}\left(\frac{\mathrm{D}}{n}-\frac{m}{n}-1\right)e^{n\theta}u=0,$$

in which $e^\theta=x$, $\mathrm{D}=\frac{d}{d\theta}$, and the notation

$$[a]^b=a(a-1)(a-2)..(a-b+1)$$

is adopted.

Secondly, I shall show that for particular values of m, the above equation admits of an immediate first integral, constituting a differential equation of the $n-1$th order, and that the results obtained by Mr. HARLEY are particular cases of this depressed equation, their difference of form arising from difference of determination of the arbitrary constant.

Thirdly, I shall solve the general differential equation by definite integrals.

Fourthly, I shall determine the arbitrary constants of the solution so as to express the mth power of that real root of the proposed algebraic equation which reduces to 1 when $x=0$.

The differential equation which forms the chief subject of these investigations certainly occupies an important place, if not one of exclusive importance, in the theory of that large class of differential equations of which the type is expressed in (3). At present, I am not aware of the existence of any differential equations of that particular type which admit of finite solution at all, otherwise than by an ultimate reduction to the form in question, or by a resolution into linear equations of the first order. It constitutes, in fact, a generalization of the form

$$u+\frac{a(\mathrm{D}-2)^2\pm n^2}{\mathrm{D}(\mathrm{D}-1)}e^{2\theta}u=0$$

given in my memoir "On a General Method in Analysis" (Philosophical Transactions for 1844, Part II.).

Formation of the Differential Equation.—General finite integral.

2. PROPOSITION.—*If* u *represent the* mth *power of any root of the algebraic equation*

$$y^n - xy^{n-1} - 1 = 0,$$

then u, *considered as a function of* x, *satisfies the linear differential equation*

$$[\mathrm{D}]^n u + \left[\frac{n-1}{n}\mathrm{D} + \frac{m}{n} - 1\right]^{n-1}\left(\frac{\mathrm{D}}{n} - \frac{m}{n} - 1\right) e^{n\theta} u = 0,$$

in which $e^\theta = x$, *and* $\mathrm{D} = \frac{d}{d\theta}$.

And the complete integral of the above differential equation will be

$$u = C_1 y_1^m + C_2 y_2^m \ldots + C_n y_n^m,$$

$y_1, y_2, \ldots y_n$ *being the* n *roots of the given algebraic equation.*

Representing y^n by z, we may give to the proposed algebraic equation the form

$$z = b + xz^{\frac{n-1}{n}}, \quad \ldots \ldots \quad (1)$$

in which $b=1$. Hence by LAGRANGE'S theorem

$$u = z^{\frac{m}{n}} = b^{\frac{m}{n}} + b^{\frac{n-1}{n}} \frac{d}{db}\left(b^{\frac{m}{n}}\right)x + \frac{d}{db}\left(b^{\frac{2(n-1)}{n}} \frac{d}{db} b^{\frac{m}{n}}\right)\frac{x^2}{1.2} + \&c., \quad \ldots \quad (2)$$

the general term of the expansion being

$$\left(\frac{d}{db}\right)^{r-1}\left\{b^{\frac{r(n-1)}{n}} \frac{d}{db} b^{\frac{m}{n}}\right\}\frac{x^r}{1.2..r}, \quad \ldots \ldots \quad (3)$$

which, on effecting the operations indicated, becomes

$$\frac{m\left[\frac{m+r(n-1)}{n} - 1\right]^{r-1} b^{\frac{m-r}{n}}}{n[r]^r} x^r \quad \ldots \ldots \quad (4)$$

We see then that u is expanded in a series of the form

$$u_0 + u_1 x + u_2 x^2 + \&c. \ ad\ inf.,$$

in which, since $b=1$,

$$u_r = \frac{m\left[\frac{m+(n-1)r}{n} - 1\right]^{r-1} \times (1)^{\frac{m-r}{n}}}{n[r]^r}; \quad \ldots \ldots \quad (5)$$

and this expression will represent the first term as well as the succeeding coefficients of the Lagrangian development, provided that we interpret the form $[p]^0$ by 1, and $[p]^{-1}$ by $\frac{1}{p+1}$.

As $1^{\frac{1}{n}}$ admits of n distinct values, the above development may be made to represent the mth power of any one of the n roots of the given algebraic equation. In particular, if we give to $1^{\frac{1}{n}}$ the particular value 1, we have

$$u_r = \frac{m\left[\frac{m+(n-1)r}{n}-1\right]^{r-1}}{n[r]^r},$$

and the expansion then represents the mth power of that particular root which, when $x=0$, reduces to 1. The law of the series upon which the formation of the differential equation depends is, as we shall perceive, independent of these determinations.

Changing r into $r-n$, we have

$$u_{r-n} = \frac{m\left[\frac{m+(n-1)r}{n}-n\right]^{r-n-1}\times 1^{\frac{m-r}{n}+1}}{n[r-n]^{r-n}},$$

whence the law of the series is seen to be

$$[r]^r u_r + \left[\frac{m+(n-1)r}{n}-1\right]^{n-1}\left(\frac{r}{n}-\frac{m}{n}-1\right)u_{n-r}=0, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and therefore, by what is shown in my memoir "On a General Method in Analysis," the differential equation defining u will be

$$[\mathrm{D}]^n u + \left[\frac{m+(n-1)\mathrm{D}}{n}-1\right]^{n-1}\left(\frac{\mathrm{D}}{n}-\frac{m}{n}-1\right)e^{n\theta}u=0, \quad . \quad . \quad . \quad . \quad (\mathrm{I})$$

in which $e^\theta = x$ and $\mathrm{D}=\frac{d}{d\theta}$.

3. As u may here represent the mth power of *any* of the roots of the given equation, it is evident that the general integral of the above differential equation will be

$$u = \mathrm{C}_1 y_1^m + \mathrm{C}_2 y_2^m \ldots + \mathrm{C}_n y_n^m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

exception arising, however, in the case in which for a particular value of m the n particular integrals $y_1^m, y_2^m, \ldots y_n^m$ cease to be independent. In such cases the above value of u constitutes an integral, but not the general integral of the differential equation.

For instance, if $m=-1$, and if we reduce the given algebraic equation to the form

$$(y^{-1})^n + xy^{-1} - 1 = 0,$$

it is evident that, except when $n=2$, we shall have

$$y_1^{-1} + y_2^{-1} \ldots + y_n^{-1} = 0.$$

Here then

$$u = \mathrm{C}_1 y_1^{-1} + \mathrm{C}_2 y_2^{-1} \ldots + \mathrm{C}_n y_n^{-1}$$

may be reduced to the form

$$u = (\mathrm{C}_1 - \mathrm{C}_n) y_1^{-1} + (\mathrm{C}_2 - \mathrm{C}_n) y_2^{-1} \ldots + (\mathrm{C}_{n-1} - \mathrm{C}_n) y_{n-1}^{-1},$$

virtually involving but $n-1$ arbitrary constants.

Such cases of failure may, however, be treated by giving to the integral a form which for the particular value of m shall become indeterminate, and then seeking the limiting value. In the above example we may write

$$u=C_1y_1^m\,..+C_{n-1}y_{n-1}^m+C_n\frac{y_1^m+y_2^m\,..+y_n^m}{m+1},$$

the last term of which becomes a vanishing fraction when $m=-1$. The true limiting form is seen to be

$$u=C_1y_1^{-1}\,..+C_{n-1}y_{n-1}^{-1}+C_n(y_1^{-1}\log y_1\,..+y_n^{-1}\log y_n). \quad . \quad . \quad . \quad . \quad (8)$$

This is the complete integral of (I.) when $m=-1$.

4. The theory of these failing cases may be viewed also in another aspect. When

$$u=C_1y_1^m+C_2y_2^m\,..+C_ny_n^m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

is an integral, but not the general integral of the differential equation (I), it must be the general integral of a differential equation involved in (I), but of a lower order. We may in fact conclude that such reduced differential equation will be deducible from the higher one by a process of integration. Let us apply this consideration to the foregoing example.

When $m=-1$, the equation (I) becomes

$$D(D-1)\,..(D-n+1)u+\frac{1}{n}\Big[\frac{n-1}{n}D-\frac{1}{n}-1\Big]^{n-1}(D-n+1)e^{n\theta}u=0.$$

Hence operating on both members with $(D-n+1)^{-1}$, we have

$$D(D-1)\,..(D-n+2)u+\frac{1}{n}\Big[\frac{n-1}{n}D-\frac{1}{n}-1\Big]^{n-1}e^{n\theta}u=Ce^{(n-1)\theta}.$$

It must then be possible to determine C so as to cause this differential equation to be satisfied by (9). First let us seek to determine C so as to cause the equation to admit of any of the particular integrals $y_1^m, y_2^m, .. y_n^m$. Substituting for u the Lagrangian expansion reduced by making $m=-1$, and giving to b any of the particular values included in the form $1^{\frac{1}{n}}$, we shall, on equating coefficients, find

$$C=\frac{-[n-3]^{n-2}}{n},$$

whence it appears that if n be greater than 2, $C=0$. Thus the reduced differential equation becomes

$$[D]^{n-1}u+\frac{1}{n}\Big[\frac{n-1}{n}D-\frac{1}{n}-1\Big]^{n-1}e^{n\theta}u=0; \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and this, when n is greater than 2, is satisfied by

$$u=C_1y_1^{-1}+C_2y_2^{-1}\,..+C_ny_n^{-1},$$

which in effect contains $n-1$ arbitrary constants, and so constitutes the complete integral of the differential equation.

If $n=2$, the differential equation becomes

$$Du+\tfrac{1}{2}(\tfrac{1}{2}D-\tfrac{3}{2})e^{2\theta}u=\frac{-1}{2}e^{\theta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

which is satisfied by $u=y_1^{-1}$ and by $u=y_2^{-1}$, but, as is evident from its unhomogeneous form, not by $u=C_1y_1^{-1}+C_2y_2^{-1}$. In this case, in fact, the condition $y_1^{-1}+y_2^{-1}=0$ not being fulfilled, the primary differential equation (I) suffers no change in the form of its general solution.

Mr. HARLEY's results are in effect transformations of (10) and (11). Since $u=y^{-1}$, it is seen that u will satisfy the algebraic equation

$$u^n+xu-1=0.$$

Transform this by assuming

$$x=-n(1-n)^{\frac{1-n}{n}}x'^{\frac{1-n}{n}},\quad u=(1-n)^{-\frac{1}{n}}x'^{-\frac{1}{n}}u',$$

and we have

$$u'^n-nu'+(n-1)x'=0,$$

which is Mr. HARLEY's algebraic equation in form. Hence, if $x'=e^{\theta'}$ and $D'=\frac{d}{d\theta'}$, we shall have

$$e^{\theta}=-n(1-n)^{\frac{1-n}{n}}e^{\frac{1-n}{n}\theta'},\quad u=(1-n)^{-\frac{1}{n}}e^{-\frac{1}{n}\theta'}u',\quad D=\frac{n}{1-n}D'.$$

And (10) will become

$$\left[\frac{nD'}{1-n}\right]^{n-1}e^{-\frac{1}{n}\theta'}u'+\frac{1}{n}\left[-D'-\frac{1}{n}-1\right]^{n-1}(-n)^n(1-n)^{1-n}e^{\left(1-n-\frac{1}{n}\right)\theta'}u'=0.$$

Multiply by $e^{\left(n-1+\frac{1}{n}\right)\theta'}$, and we have

$$\left[\frac{n\left(D'-n+1-\frac{1}{n}\right)}{1-n}\right]^{n-1}e^{(n-1)\theta'}u'-\left[-D'+n-2\right]^{n-1}(-n)^{n-1}(1-n)^{1-n}u'=0.$$

Now

$$\left[\frac{n\left(D'-n+1-\frac{1}{n}\right)}{1-n}\right]^{n-1}=\left[\frac{n}{1-n}D'+n-\frac{1}{1-n}\right]^{n-1}=(-1)^{n-1}\left[\frac{n}{n-1}D'-\frac{2n-1}{n-1}\right]^{n-1},$$

and

$$[-D'+n-2]^{n-1}=(-1)^{n-1}[D']^{n-1}.$$

Hence

$$\left[\frac{n}{n-1}D'-\frac{2n-1}{n-1}\right]^{n-1}e^{(n-1)\theta'}u'-\left(\frac{n}{n-1}\right)^{n-1}[D']^{n-1}u'=0,$$

or

$$[D']^{n-1}u'-\left(\frac{n-1}{n}\right)^{n-1}\left[\frac{n}{n-1}D'-\frac{2n-1}{n-1}\right]^{n-1}e^{(n-1)\theta'}u'=0,$$

which is Mr. HARLEY's equation (1), art. 1. When $n=2$, we obtain from (11), by the same transformations, Mr. HARLEY's second equation (2), art. 1.

Not only for the particular value $m=-1$, but apparently for all integer values of m, the general differential equation (I) admits of one integration. It may be said that while the differential equation determining the form of the mth power of a root of the algebraic equation is in general of the nth order, this equation may, when m is an integer, be reduced to an equation of the $n-1$th order; not, however, like the higher equation, unvarying in its type. I have thus verified some other particular forms obtained by Mr. HARLEY.

Solution of the Differential Equation by Definite Integrals.

5. On account of the difficulty of the investigation, I propose to employ two distinct methods leading to coincident results.

First Method.—Operating on both sides of the given differential equation (I) with $\{[D]^n\}^{-1}$, we have

$$u+\frac{\left[\frac{n-1}{n}D+\frac{m}{n}-1\right]^{n-1}\left(\frac{D}{n}-\frac{m}{n}-1\right)}{[D]^n}e^{n\theta}u=C_0+C_1e^{\theta}\cdot\cdot+C_{n-1}e^{(n-1)\theta}, \quad . \quad . \quad . \quad (1)$$

C_0, C_1, .. C_{n-1} being arbitrary constants. Let us represent

$$\frac{\left[\frac{n-1}{n}D+\frac{m}{n}-1\right]^{n-1}\left(\frac{D}{n}-\frac{m}{n}-1\right)}{[D]^n}e^{n\theta}U,$$

whatever the nature of the subject U, by ϱU, then the differential equation becomes

$$u+\varrho\, u=C_0+C_1e^{\theta}\cdot\cdot+C_{n-1}e^{(n-1)\theta},$$

or

$$(1+\varrho)u=C_0+C_1e^{\theta}\cdot\cdot+C_{n-1}e^{(n-1)\theta};$$

$$\therefore\quad u=(1+\varrho)^{-1}\{C_0+C_1e^{\theta}\cdot\cdot+C_{n-1}e^{(n-1)\theta}\}=\Sigma_iC_i(1+\varrho)^{-1}e^{i\theta},$$

the summation extending from $i=0$ to $i=n-1$.

Now

$$(1+\varrho)^{-1}e^{i\theta}=(1-\varrho+\varrho^2-\varrho^3\,..)e^{i\theta}.$$

But if

$$\phi(D)=\frac{\left[\frac{n-1}{n}D+\frac{m}{n}-1\right]^{n-1}\left(\frac{D}{n}-\frac{m}{n}-1\right)}{[D]^n},$$

we have

$$\begin{aligned}\varrho e^{i\theta}&=\phi(D)e^{n\theta}e^{i\theta},\\ \varrho^2e^{i\theta}&=\phi(D)e^{n\theta}\phi(D)e^{(n+i)\theta}=\phi(D)\phi(D-n)e^{(2n+i)\theta},\\ \varrho^pe^{i\theta}&=\phi(D)\phi(D-n)\,..\,\phi\big(D-(p-1)n\big)e^{(pn+i)\theta}.\end{aligned}$$

But from the form of $\phi(D)$ it is easily seen that

$$\phi(D)\phi(D-n)=\frac{\left[\frac{n-1}{n}D+\frac{m}{n}-1\right]^{2(n-1)}\left[\frac{D}{n}-\frac{m}{n}-1\right]^2}{[D]^{2n}},$$

and generally

$$\phi(D)\phi(D-n)\,..\,\phi\big(D-(p-1)n\big)=\frac{\left[\frac{n-1}{n}D+\frac{m}{n}-1\right]^{p(n-1)}\left[\frac{D}{n}-\frac{m}{n}-1\right]^p}{[D]^{pn}};$$

$$\therefore\ \varrho^pe^{i\theta}=\frac{\left[\frac{n-1}{n}D+\frac{m}{n}-1\right]^{p(n-1)}\left[\frac{D}{n}-\frac{m}{n}-1\right]^p}{[D]^{pn}}e^{(pn+i)\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now

$$[a]^b=a(a-1)..(a-b+1)$$
$$=\frac{\Gamma(a+1)}{\Gamma(a-b+1)},$$

provided that $a+1$ and $a-b+1$ are positive. This law we can extend symbolically to expressions in which D appears, provided that, in the application of the symbolic forms thence arising, D shall admit of an interpretation which shall effectively make the subjects of the symbol Γ to be positive numerical magnitudes. Under this condition we have then

$$\varrho^p e^{i\theta}=\frac{\dfrac{\Gamma\left(\frac{n-1}{n}\mathrm{D}+\frac{m}{n}\right)}{\Gamma\left(\frac{n-1}{n}\mathrm{D}+\frac{m}{n}-p(n-1)\right)}\times\dfrac{\Gamma\left(\frac{\mathrm{D}}{n}-\frac{m}{n}\right)}{\Gamma\left(\frac{\mathrm{D}}{n}-\frac{m}{n}-p\right)}}{\dfrac{\Gamma(\mathrm{D}+1)}{\Gamma(\mathrm{D}-pn+1)}}e^{(pn+i)\theta}$$

$$=\Phi(\mathrm{D})\Psi(\mathrm{D})e^{(pn+i)\theta},$$

where

$$\Phi(\mathrm{D})=\frac{\Gamma\left(\frac{n-1}{n}\mathrm{D}+\frac{m}{n}\right)\Gamma\left(\frac{\mathrm{D}}{n}-\frac{m}{n}\right)}{\Gamma(\mathrm{D}+1)},$$

$$\Psi(\mathrm{D})=\frac{\Gamma(\mathrm{D}-pn+1)}{\Gamma\left(\frac{n-1}{n}\mathrm{D}+\frac{m}{n}-p(n-1)\right)\Gamma\left(\frac{\mathrm{D}}{n}-\frac{m}{n}-p\right)}.$$

Now

$$\Psi(\mathrm{D})e^{(pn+i)\theta}=\Psi(pn+i)e^{(pn+i)\theta}$$
$$=\frac{\Gamma(i+1)}{\Gamma\left(\frac{n-1}{n}i+\frac{m}{n}\right)\Gamma\left(\frac{i-m}{n}\right)}e^{(pn+i)\theta}.$$

We see then that the conditions

$$(n-1)i+m\gtreqless 0,\quad i-m\gtreqless 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

must be satisfied. For $i=0$ these conditions are inconsistent, and the proposed employment of Γ therefore unlawful. For values of i greater than 0 the conditions will be found to amount to this, viz. that m must lie between the limits $-(n-1)$ and 1. We shall suppose m thus conditioned, and shall consider first the case in which $i>0$.

Here then we have, interpreting D by $pn+i$ in $\Psi(\mathrm{D})$, but leaving it unchanged in $\Phi(\mathrm{D})$,

$$\varrho^p e^{i\theta}=\frac{\Gamma\left(\frac{n-1}{n}\mathrm{D}+\frac{m}{n}\right)\Gamma\left(\frac{\mathrm{D}}{n}-\frac{m}{n}\right)}{\Gamma(\mathrm{D}+1)}\quad\frac{\Gamma(i+1)}{\Gamma\left(\frac{n-1}{n}i+\frac{m}{n}\right)\Gamma\left(\frac{i-m}{n}\right)}e^{(pn+i)\theta}, \quad . \quad . \quad . \quad . \quad (4)$$

it being seen that if we similarly interpreted D in $\Phi(\mathrm{D})$ the conditions relative to Γ would be satisfied throughout.

Hence if we write

$$\frac{\Gamma(i+1)C_i}{\Gamma\left(\frac{n-1}{n}i+\frac{m}{n}\right)\Gamma\left(\frac{i-m}{n}\right)}=A_i,$$

we shall have

$$(1-\varrho+\varrho^2-\varrho^3+\&c.)C_ie^{i\theta}=\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D+1)}A_i(e^{i\theta}-e^{(n+i)\theta}+e^{(2n+i)\theta}-\&c.)$$

$$=\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D+1)}\frac{A_ie^{i\theta}}{1+e^{n}},$$

and therefore

$$\Sigma_iC_i(1+\varrho)^{-1}e^{i\theta}=\Sigma_i\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D+1)}\frac{A_ie^{i\theta}}{1+e^{n\theta}}, \quad . \ . \ . \ . \ . \ . \ . \quad (5)$$

the summation extending from $i=1$ to $i=n-1$.

Consider next the case in which $i=0$. We have, when p is not less than 1,

$$\varrho^pC_0=\varrho^{p-1}\varrho C_0$$
$$=\varrho^{p-1}\phi(D)e^{n\theta}C_0$$
$$=C_0\phi(n)\varrho^{p-1}e^{n\theta}.$$

But changing in (4) p into $p-1$, and i into n,

$$\varrho^{p-1}e^{i\theta}=\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D+1)}\frac{\Gamma(n+1)}{\Gamma\left(n-1+\frac{m}{n}\right)\Gamma\left(\frac{n-m}{n}\right)}e^{pn\theta}.$$

Hence, if we write

$$C_0\phi(n)\frac{\Gamma(n+1)}{\Gamma\left(n-1+\frac{m}{n}\right)\Gamma\left(\frac{n-m}{n}\right)}=-A_n, \quad . \ . \ . \ . \ . \ . \ . \quad (5')$$

we have for all positive integral values of p,

$$\varrho^pC_0=-A_n\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D+1)}e^{pn\theta},$$

and therefore

$$(1-\varrho+\varrho^2-\varrho^3+\&c.)C_0=C_0+\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D+1)}(A_ne^{n\theta}-A_ne^{2n\theta}+A_ne^{3n\theta}-\&c.);$$

$$\therefore\quad(1+\varrho)^{-1}C_0=C_0+\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D+1)}\frac{A_ne^{n\theta}}{1+e^{n\theta}}.$$

Combining this with (5), we find

$$u=C_0+\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D+1)}\left\{\frac{A_1e^{\theta}+A_2e^{2\theta}\ldots+A_ne^{n\theta}}{1+e^{n\theta}}\right\}$$

$$=C_0+\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D)}D^{-1}\left\{\frac{A_1e^{\theta}+A_2e^{2\theta}\ldots+A_ne^{n\theta}}{1+e^{n\theta}}\right\}.$$

Now, resolving the rational fraction, we have

$$D^{-1}\frac{A_1e^{\theta}+A_2e^{2\theta}\ldots+A_ne^{n\theta}}{1+e^{n\theta}}=D^{-1}\left\{\frac{N_1e^{\theta}}{1-\alpha_1e^{\theta}}+\frac{N_2e^{\theta}}{1-\alpha_2e^{\theta}}\ldots+\frac{N_ne^{\theta}}{1-\alpha_ne^{\theta}}\right\}$$

$$=B_1\log(1-\alpha_1e^{\theta})+B_2\log(1-\alpha_2e^{\theta})\ldots+B_n\log(1-\alpha_ne^{\theta}),$$

$\alpha_1, \alpha_2, \ldots \alpha_n$ being the nth roots of -1, and $B_i=\frac{-N_i}{\alpha_i}$. Hence

$$u=C_0+\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D)}\{B_1\log(1-\alpha_1e^{\theta})\ldots+B_n\log(1-\alpha_ne^{\theta})\}. \quad . \quad . \quad (6)$$

In this expression $B_1, \ldots B_n$, being generated from the arbitrary constants $C_0, C_1, \ldots C_{n-1}$, may themselves be regarded as arbitrary constants. And this being done, C_0 will become a dependent constant, the form of which it will be necessary to determine.

First, however, let us endeavour to interpret by a definite integral the symbolic function of D.

We know that a and b being positive quantities,

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int_0^1 dt\ t^{a-1}(1-t)^{b-1}=\int_0^{\infty}\frac{dt\ t^{a-1}}{(1+t)^{a+b}}.$$

If we employ the second of these forms, we shall have

$$\frac{\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)}{\Gamma(D)}\varphi(e^{\theta})=\int_0^{\infty}dt\ \frac{t^{\frac{n-1}{n}D+\frac{m}{n}-1}}{(1+t)^{D}}\varphi(e^{\theta})$$

$$=\int_0^{\infty}dt\ t^{\frac{m}{n}-1}\left(\frac{t^{\frac{n-1}{n}}}{1+t}\right)^{D}\varphi(e^{\theta})$$

$$=\int_0^{\infty}dt\ t^{\frac{m}{n}-1}\varphi\left(\frac{t^{\frac{n-1}{n}}e^{\theta}}{1+t}\right)$$

by a known symbolical form of TAYLOR's theorem. Hence if

$$\frac{t^{\frac{n-1}{n}}}{1+t}=T,$$

we have

$$u=C_0+B_1\int_0^{\infty}dt\ t^{\frac{m}{n}-1}\log(1-\alpha_1Te^{\theta})\cdots+B_n\int_0^{\infty}dt\ t^{\frac{m}{n}-1}\log(1-\alpha_nTe^{\theta}). \quad . \quad . \quad (7)$$

6. In determining C_0 the following theorem will be of use, viz.:—

If r *be a positive integer, and* a *positive and less than* r, *then*

$$\Gamma(a)\Gamma(r-a)=\frac{[r-a-1]^{r-1}\pi}{\sin(a\pi)}. \quad \ldots \quad (8)$$

This may be proved as follows:—

Let i be the greatest integer in a, and let $a-i=a'$. Then

$$\Gamma(a)\Gamma(r-a)=[a-1]^i\Gamma(a')\times[r-a-1]^{r-i-1}\Gamma(1-a').$$

But a' being a positive proper fraction,

$$\Gamma(a')\Gamma(1-a')=\frac{\pi}{\sin(a'\pi)},$$

and

$$[a-1]^i=(a-1)(a-2)\ldots(a-i)$$
$$=(-1)^i(i-a)(i-a-1)\ldots(1-a),$$

$$[r-a-1]^{r-i-1}=(r-a-1)(r-a-2)\ldots(i-a+1);$$

$$\therefore \quad [r-a-1]^{r-i-1}[a-1]^i=(-1)^i(r-a-1)\ldots(i-a+1)\times(i-a)\ldots(1-a)$$
$$=(-1)^i[r-a-1]^{r-1}.$$

Hence

$$\Gamma(a)\Gamma(r-a)=(-1)^i[r-a-1]^{r-1}\frac{\pi}{\sin(a'\pi)}.$$

But

$$\sin(a'\pi)=\sin(a\pi-i\pi)=(-1)^i\sin(a\pi),$$

$$\therefore \quad \Gamma(a)\Gamma(r-a)=\frac{[r-a-1]^{r-1}\pi}{\sin(a\pi)},$$

as was to be proved.

Now in the instance before us we have by (5′)

$$C_0=-A_n\frac{\Gamma\left(n-1+\frac{m}{n}\right)\Gamma\left(\frac{n-m}{n}\right)}{\Gamma(n+1)\phi(n)},$$

where

$$\phi(n)=\frac{\left[n+\frac{m}{n}-2\right]^{n-1}\left(-\frac{m}{n}\right)}{[n]^n}.$$

Hence, since $\Gamma(n+1)=[n]^n$,

$$C_0=A_n\frac{\Gamma\left(n-1+\frac{m}{n}\right)\Gamma\left(1-\frac{m}{n}\right)}{\left[n+\frac{m}{n}-2\right]^{n-1}\times\frac{m}{n}};$$

wherefore $1-\frac{m}{n}$ being a positive quantity, and n a positive integer, we have, by the above theorem,

$$\Gamma\left(n-1+\frac{m}{n}\right)\Gamma\left(1-\frac{m}{n}\right)=\frac{\left[n-2+\frac{m}{n}\right]^{n-1}\pi}{\sin\left(1-\frac{m}{n}\right)\pi}$$

$$=\frac{\left[n-2+\frac{m}{n}\right]^{n-1}\pi}{\sin\left(\frac{m}{n}\pi\right)}.$$

Accordingly

$$C_0=\frac{A_n\pi}{\frac{m}{n}\sin\frac{m\pi}{n}}.$$

But since

$$\frac{N_1e^\theta}{1-\alpha_1e^\theta}\cdot\cdot+\frac{N_ne^\theta}{1-\alpha_ne^\theta}=\frac{A_1e^\theta\cdot\cdot+A_ne^{n\theta}}{1+e^{n\theta}},$$

we have

$$A_n=(-1)^{n-1}\left(\frac{N_1}{\alpha_1}\cdot\cdot+\frac{N_n}{\alpha_n}\right)\alpha_1\alpha_2\ldots\alpha_n$$

$$=(-1)^n\quad(B_1\cdot\cdot+B_n)\times(-1)^n$$

$$=\qquad B_1\cdot\cdot+B_n.$$

Therefore, finally,

$$C_0=\frac{B_1+B_2\cdot\cdot+B_n}{\frac{m}{n}\sin\frac{m\pi}{n}}.$$

Substituting in (7), and replacing e^θ by x, we have

$$u=\frac{(B_1+B_2\cdot\cdot+B_n)\pi}{\frac{m}{n}\sin\frac{m\pi}{n}}+B_1\int_0^\infty dt\, t^{\frac{m}{n}-1}\log(1-\alpha_1xT)\cdot\cdot+B_n\int_0^\infty dt\, t^{\frac{m}{n}-1}\log(1-\alpha_nxT),\quad .\quad (9)$$

wherein, it must be remembered, that $\alpha_1, \alpha_2, \ldots \alpha_n$ are the several nth roots of -1, and

$$T=\frac{t^{\frac{n-1}{n}}}{1+t}.$$

And this is the general integral of (I), $B_1, B_2, \ldots B_n$ being the arbitrary constants of the solution.

Second Method.—7. For the *finite* solution of differential equations of the form

$$f_0(D)u+f_1(D)e^{n\theta}u=0,$$

it is usually convenient to reduce them to the form

$$u+\frac{f_1(D)}{f_0(D)}e^{n\theta}u=\{f_0(D)\}^{-1}0,$$

which falls under the general type

$$u+\phi(\mathrm{D})e^{n\theta}u=\mathrm{U}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

U being a function of θ when the inverse operation $\{f_0(\mathrm{D})\}^{-1}0$ has been performed.

The theory of equations of the above type has been discussed fully in my memoir "On a General Method in Analysis." In particular it is there shown that the above equation can be converted into another of the same type,

$$v+\psi(\mathrm{D})e^{n\theta}v=\mathrm{V},$$

by assuming

$$u=\mathrm{P}_n\frac{\phi(\mathrm{D})}{\psi(\mathrm{D})}v, \qquad \mathrm{V}=\left\{\mathrm{P}_n\frac{\phi(\mathrm{D})}{\psi(\mathrm{D})}\right\}^{-1}\mathrm{U}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where

$$\mathrm{P}_n\frac{\phi(\mathrm{D})}{\psi(\mathrm{D})}=\frac{\phi(\mathrm{D})\phi(\mathrm{D}-n)\phi(\mathrm{D}-2n)\,..\,ad\ inf.}{\psi(\mathrm{D})\psi(\mathrm{D}-n)\psi(\mathrm{D}-2n)\,..\,ad\ inf.}.$$

This theory I shall apply here, not to the ordinary finite solution, but to the solution by definite integrals of the differential equation (I). In doing this I shall give to U and V the particular values 0. We are justified in doing this by the canons relating to the arbitrary constants which are laid down in the memoir; but it will suffice here to direct attention to the fact that while the processes employed are strictly speaking particular, they lead to a solution involving the requisite number of arbitrary constants, and at the same time of the proper *form*, as manifested by the succession of the indices in its development.

Giving then to (I) the form

$$u+\frac{\left[\frac{n-1}{n}\mathrm{D}+\frac{m}{n}-1\right]^{n-1}\left(\frac{\mathrm{D}}{n}-\frac{m}{n}-1\right)}{[\mathrm{D}]^n}e^{n\theta}u=0,$$

assume as the transformed equation

$$v+\frac{1}{[\mathrm{D}]^n}e^{n\theta}v=0.$$

Then by (2)

$$u=\mathrm{P}_n\left\{\left[\frac{n-1}{n}\mathrm{D}+\frac{m}{n}-1\right]\left(\frac{\mathrm{D}}{n}-\frac{m}{n}-1\right)\right\}v.$$

Now

$$\mathrm{P}_n\left[\frac{n-1}{n}\mathrm{D}+\frac{m}{n}-1\right]^{n-1}=\left(\frac{n-1}{n}\mathrm{D}+\frac{m}{n}-1\right)\left(\frac{n-1}{n}\mathrm{D}+\frac{m}{n}-2\right)..\ ad\ inf.,$$

since representing $\left[\frac{n-1}{n}\mathrm{D}+\frac{m}{n}-1\right]^{n-1}$ by $\phi(\mathrm{D})$, the first term in the factorial expression of $\phi(\mathrm{D}-n)$ will so follow the last term in that of $\phi(\mathrm{D})$ as to leave the law of factorial succession unbroken. Again, if $\mathrm{A}_ie^{i\theta}$ be any term in the development of v, we have, i being a positive integer,

$$\left(\frac{n-1}{n}\mathrm{D}+\frac{m}{n}-1\right)\left(\frac{n-1}{n}\mathrm{D}+\frac{m}{n}-2\right)..\mathrm{A}_ie^{i\theta}$$

$$=\mathrm{A}_i\left(\frac{n-1}{n}i+\frac{m}{n}-1\right)\left(\frac{n-1}{n}i+\frac{m}{n}-2\right)..e^{i\theta}$$

$$=\mathrm{A}_i\mathrm{C}\Gamma\left(\frac{n-1}{n}i+\frac{m}{n}\right)e^{i\theta},$$

C being a constant, the value of which does not change with i. Hence we may write

$$P_n\left[\frac{n-1}{n}D+\frac{m}{n}-1\right]^{n-1}=C\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right),$$

and in like manner

$$P_n\left(\frac{D}{n}-\frac{m}{n}-1\right)=C'\Gamma\left(\frac{D}{n}-\frac{m}{n}\right).$$

The legitimacy of the introduction of Γ depends upon the condition

$$\frac{n-1}{n}i+\frac{m}{n}>0,\ \frac{i}{n}-\frac{m}{n}>0,$$

so that the value $i=0$ is inadmissible, as we have already assumed. Moreover m must lie between the limits $-(n-1)$ and 1.

Since $e^\theta=x$, the equation for v is equivalent to

$$\frac{d^nv}{dx^n}+v=0,$$

whence

$$v=c_1e^{\alpha_1x}+c_2e^{\alpha_2x}..+c_ne^{\alpha_nx},$$

in which $\alpha_1, \alpha_2, .. \alpha_n$ are the nth roots of -1. This value of v can be expanded in ascending powers of x in the form

$$\begin{aligned}v&=v_0+v_1x+v_2x^2+\&c.\\ &=v_0+v_1e^\theta+v_2e^{2\theta}+\&c.\end{aligned}$$

Hence $u-u_0$ representing that part of u which contains positive and integral powers of x, we shall have

$$u-u_0=CC'\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)(v-v_0).$$

Now
$$\begin{aligned}v-v_0&=C_1(e^{\alpha_1x}-1)+C_2(e^{\alpha_2x}-1)..+C_n(e^{\alpha_nx}-1)\\ &=\Sigma C_i(e^{\alpha_ix}-1),\end{aligned}$$

the summation extending from $i=1$ to $i=n$. Hence, merging CC' in the arbitrary constants $C_1, .. C_n$, we have

$$u=u_0+\Sigma C_i\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)(e^{\alpha_ix}-1), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which $x=e^\theta$. This expression we now propose to interpret by definite integrals.

Now
$$e^{\alpha_ix}-1=\int_0^x\alpha_ie^{\alpha_ih}dh.$$

Substituting and merging α_i in the arbitrary constant C_i, we have

$$\begin{aligned}u&=u_0+\Sigma C_i\Gamma\left(\frac{n-1}{n}D+\frac{m}{n}\right)\Gamma\left(\frac{D}{n}-\frac{m}{n}\right)\int_0^x e^{\alpha_ih}dh\\ &=u_0+\Sigma C_i\int_0^\infty dse^{-s}s^{\frac{n-1}{n}D+\frac{m}{n}-1}\int_0^\infty dte^{-t}t^{\frac{D}{n}-\frac{m}{n}-1}\int_0^x e^{\alpha_i}\,dh\end{aligned}$$

on interpreting the Γ functions in the usual manner. We may therefore write

$$u=u_0+\Sigma C_i\int_0^\infty\int_0^\infty dsdte^{-(s+t)}s^{\frac{m}{n}-1}t^{-\frac{m}{n}-1}\int_0^{xs^{\frac{n-1}{n}}t^{\frac{1}{n}}}e^{\alpha_i h}\,dh,$$

since by the symbolical form of Taylor's theorem

$$s^{\frac{n-1}{n}\mathrm{D}}t^{\frac{\mathrm{D}}{n}}\varphi(x)=\left(s^{\frac{n-1}{n}}t^{\frac{1}{n}}\right)^{\mathrm{D}}\varphi(x)=\varphi\left(xs^{\frac{n-1}{n}}t^{\frac{1}{n}}\right).$$

Let us now transform the double integral relative to s and t by assuming

$$s=vt,$$

and making v and t the new system of variables. We shall have

$$dsdt=tdvdt,$$

while the limits of v and t will be 0 and ∞. Hence

$$u=u_0+\Sigma C_i\int_0^\infty\int_0^\infty dvdte^{-(1+v)t}v^{\frac{m}{n}-1}t^{-1}\int_0^{xv^{\frac{n-1}{n}}t}e^{\alpha_i h}\,dh.$$

Again, transform the double integral relative to t and h, by assuming $h=ty$. We shall have $dh=tdy$, and the limits of y will be 0 and $xv^{\frac{n-1}{n}}$. Whence

$$u=u_0+\Sigma C_i\int_0^\infty\int_0^\infty\int_0^{xv^{\frac{n-1}{n}}}dvdtdye^{-(1+v-\alpha_i y)t}v^{\frac{m}{n}-1}.$$

Integrating with respect to t, we have

$$u=u_0+\Sigma C_i\int_0^\infty dv\int_0^{xv^{\frac{n-1}{n}}}dy\,\frac{v^{\frac{m}{n}-1}}{1+v-\alpha_i y}.$$

Now integrating with respect to y, and merging $\frac{-1}{\alpha_i}$ in the arbitrary constants,

$$\left.\begin{aligned}u&=u_0+\Sigma C_i\int_0^\infty dv\,v^{\frac{m}{n}-1}\left\{\log\left(1+v-\alpha_i xv^{\frac{n-1}{n}}\right)-\log(1+v)\right\}\\&=u_0+\Sigma C_i\int_0^\infty dv\,v^{\frac{m}{n}-1}\log\left(1-\frac{\alpha_i xv^{\frac{n-1}{n}}}{1+v}\right)\end{aligned}\right\}.\quad\ldots\quad(4)$$

It remains to determine u_0.

Developing the function under the sign of integration in ascending powers of x, and effecting the integration for each term separately, we find, for the coefficient of x^n, the expression

$$u_n=\Sigma C_i\frac{\Gamma\left(\frac{m}{n}+n-1\right)\Gamma\left(1-\frac{m}{n}\right)}{n\Gamma(n)};$$

but from the law of the series as expressed in (6), art. 2,

$$u_n=-\frac{\left[\frac{m}{n}+n-2\right]^{n-1}\times\left(-\frac{m}{n}\right)}{[n]^n}u_0.$$

Equating these values,

$$u_0=\Sigma C_i\frac{\Gamma\left(\frac{m}{n}+n-1\right)\Gamma\left(1-\frac{m}{n}\right)}{\frac{m}{n}\left[\frac{m}{n}+n-2\right]^{n-1}}$$

$$=\Sigma C_i\frac{\pi}{\frac{m}{n}\sin\frac{m\pi}{n}}$$

by the reductions of art. 6.

Hence, finally,

$$u=\Sigma\frac{C_i\pi}{\frac{m}{n}\sin\frac{m\pi}{n}}+\Sigma C_i\int_0^\infty dv\ v^{\frac{m}{n}-1}\log\left(1-\alpha_i\frac{xv^{\frac{n-1}{n}}}{1+v}\right), \quad . \quad . \quad . \quad . \quad . \quad \text{(II)}$$

which agrees with the previous result.

Determination of the Constants.

8. I propose here to determine the constants of the general integral (II), so as to obtain an expression for the mth power of that particular (real) root of the equation

$$y^n-xy^{n-1}-1=0$$

which becomes unity when $x=0$.

We have

$$u=\Sigma C_i\frac{\pi}{\frac{m}{n}\sin\frac{m\pi}{n}}+\Sigma C_i\int_0^\infty dv\ v^{\frac{m}{n}-1}\log(1-\alpha_i xV), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $V=\frac{v^{\frac{n-1}{n}}}{1+v}$, and α_i represents in succession the different nth roots of -1.

The coefficient of x^r in the expansion of this value of u in ascending powers of x will be found to be

$$-\Sigma C_i\alpha_i^r\frac{\Gamma\left(\frac{m+(n-1)r}{n}\right)\Gamma\left(\frac{r-m}{n}\right)}{r\Gamma(r)},$$

and its coefficient in the expansion of y^m by LAGRANGE's theorem is, for the particular root in question,

$$\frac{m\left[\frac{m+(n-1)r}{n}-1\right]^{r-1}}{n[r]^r},$$

equating which we have

$$\Sigma C_i\alpha_i^r=-\frac{m\left[\frac{m+(n-1)r}{n}-1\right]^{r-1}}{n\Gamma\left(\frac{m+(n-1)r}{n}\right)\Gamma\left(\frac{r-m}{n}\right)}.$$

But by the theorem of art. 6,

$$\Gamma\left(\frac{m+(n-1)r}{n}\right)\Gamma\left(\frac{r-m}{n}\right)=\Gamma\left(\frac{r-m}{n}\right)\Gamma\left(r-\frac{r-m}{n}\right)$$

$$=\left[r-\frac{r-m}{n}-1\right]^{n-1}\frac{\pi}{\sin\left(\frac{r-m}{n}\pi\right)}$$

$$=\frac{\left[\frac{m+(n-1)r}{n}-1\right]^{r-1}\pi}{\sin\left(\frac{m-r}{n}\pi\right)}.$$

Hence

$$\Sigma_i C_i \alpha_i^r = -\frac{m\sin\left(\frac{r-m}{n}\pi\right)}{n\pi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Giving, in this equation, to r any particular system of n values, we shall obtain a system of n linear equations for the determination of the n constants C_1, C_2, .. C_n. We shall form this system by giving to r the values 1, 2, .. n.

Now α_j representing any *particular* root selected from the series α_1, α_2, .. α_n, multiply the above typical equation by α_j^{n-r}, and then, giving to r the successive values 1, 2 .. n, form the sum of the equations thus arising. The result may be expressed in the form

$$\Sigma_i C_i \Sigma_r \alpha_i^r \alpha_j^{n-r} = -\frac{m}{n\pi}\Sigma_r \sin\left(\frac{r-m}{n}\pi\right)\alpha_j^{n-r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the summations with respect to i and r being both from 1 to n inclusive.

But

$$\Sigma_r \alpha_i^r \alpha_j^{n-r} = \alpha_i \alpha_j^{n-1} + \alpha_i^2 \alpha_j^{n-2} .. + \alpha_i^n$$

$$= \alpha_i(\alpha_j^{n-1} + \alpha_i \alpha_j^{n-2} .. + \alpha_i^{n-1})$$

$$= \alpha_i \frac{\alpha_j^n - \alpha_i^n}{\alpha_j - \alpha_i}.$$

Now when α_i is not equal to α_j, this expression vanishes, since $\alpha_i^n = \alpha_j^n = -1$. When, however, $\alpha_i = \alpha_j$, the fraction $\frac{\alpha_j^n - \alpha_i^n}{\alpha_j - \alpha_i}$ becomes indeterminate, and its true limiting value is seen to be $n\alpha_j^n = -n$. Hence (3) becomes

$$-nC_j = -\frac{m}{n\pi}\Sigma_r \sin\left(\frac{r-m}{n}\pi\right)\alpha_j^{n-r},$$

$$\therefore \quad C_j = \frac{m}{n^2\pi}\Sigma_r \sin\left(\frac{r-m}{n}\pi\right)\alpha_j^{n-r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We have thus solved the linear system of equations. We have still to reduce this solution by effecting the summation in the second member.

Now to α_j we may give the form $e^{\frac{2j+1}{n}\pi\sqrt{-1}}$, which will represent all the nth roots of -1 in succession if we give to j the series of values 1, 2, .. n. Hence substituting for α_j

the above value, and giving to $\sin\left(\frac{m-r}{n}\pi\right)$ its exponential form, we have

$$\Sigma_r \sin\left(\frac{m-r}{n}\pi\right)\alpha_j^{-r}=\Sigma_r\frac{e^{\frac{m-r-(2j+1)r}{n}\pi\sqrt{-1}}-e^{\frac{-(m-r)-(2j+1)r}{n}\pi\sqrt{-1}}}{2\sqrt{-1}}$$

$$=\frac{e^{\frac{m\pi}{n}\sqrt{-1}}\Sigma_r e^{\frac{-2(j+1)r\pi}{n}\sqrt{-1}}-e^{\frac{m-\pi}{n}\sqrt{-1}}\Sigma_r e^{\frac{-2jr\pi}{n}\sqrt{-1}}}{2\sqrt{-1}}.$$

Now in general

$$\Sigma_r e^{kr\pi\sqrt{-1}}=e^{k\pi\sqrt{-1}}+e^{2k\pi\sqrt{-1}}\ldots+e^{nk\pi\sqrt{-1}}$$

$$=\frac{e^{(n+1)k\pi\sqrt{-1}}-e^{k\pi\sqrt{-1}}}{e^{k\pi\sqrt{-1}}-1}$$

$$=e^{\frac{k(n+1)\pi}{2}\sqrt{-1}}\times\frac{e^{\frac{kn\pi}{2}\sqrt{-1}}-e^{\frac{-kn\pi}{2}\sqrt{-1}}}{e^{k\frac{\pi}{2}\sqrt{-1}}-e^{\frac{-k\pi}{2}\sqrt{-1}}}$$

$$=e^{\frac{k(n+1)}{2}\pi\sqrt{-1}}\times\frac{\sin\frac{kn\pi}{2}}{\sin\frac{k\pi}{2}}.$$

Putting therefore $$k=-\frac{2(j+1)}{n},$$

we have $$\Sigma_r e^{\frac{-2(j+1)r\pi}{n}\sqrt{-1}}=e^{\frac{-(j+1)(n+1)}{n}\pi\sqrt{-1}}\frac{\sin(j+1)\pi}{\sin\frac{(j+1)\pi}{n}},$$

and putting $$k=e\frac{-2j}{n},$$

$$\Sigma_r e^{\frac{-2jr\pi}{n}\sqrt{-1}}=e^{\frac{-j(n+1)}{n}\pi\sqrt{-1}}\frac{\sin j\pi}{\sin\frac{j\pi}{n}}.$$

Hence $$\Sigma_r \sin\left(\frac{m-r}{n}\pi\right)\alpha_j^{-r}=\frac{1}{2\sqrt{-1}}\left\{\begin{array}{l} e^{\frac{m-(j+1)(n+1)}{n}\pi\sqrt{-1}}\dfrac{\sin(j+1)\pi}{\sin\frac{(j+1)\pi}{n}} \\ -e^{\frac{-m-j(n+1)}{n}\pi\sqrt{-1}}\dfrac{\sin j\pi}{\sin\frac{j\pi}{n}}\end{array}\right\}$$

Now $$\frac{\sin(j+1)\pi}{\sin\frac{(j+1)\pi}{n}}=0$$

for all values of j taken from the series 1, 2, .. n except the value $n-1$, for which the expression becomes indeterminate in form, and has for its true value

$$\frac{\pi\cos n\pi}{\frac{\pi}{n}\cos\frac{n\pi}{n}}=\frac{n\cos n\pi}{\cos\pi}=\pm n,$$

as n is odd or even.

So too
$$\frac{\sin j\pi}{\sin\frac{j\pi}{n}}=0$$

for all values of j taken from the series 1, 2, .. n except the value n, for which its true value is
$$\frac{n\cos n\pi}{\cos\pi}=\pm n$$
as n is odd or even.

Hence when j stands for any of the integers 1, 2, .. $n-2$, we have
$$\Sigma_r\sin\left(\frac{m-r}{n}\pi\right)\alpha_j^{-r}=0.$$
When $j=n-1$, we have
$$\Sigma_r\sin\left(\frac{m-r}{n}\pi\right)\alpha_j^{-r}=\pm\frac{n}{2\sqrt{-1}}e^{\frac{m-n(n+1)}{n}\pi\sqrt{-1}},$$

the upper or lower sign being taken according as n is odd or even. To the second member we may give the form
$$\pm\frac{n}{2\sqrt{-1}}e^{\frac{m\pi}{n}\sqrt{-1}}\left(\cos(n+1)\pi-\sqrt{-1}\sin(n+1)\pi\right)=\frac{n}{2\sqrt{-1}}e^{\frac{m\pi}{n}\sqrt{-1}},$$

since $\sin(n+1)\pi=0$, $\cos(n+1)\pi=\pm1$, as n is odd or even.

Thus when $j=n-1$, we have
$$\Sigma_r\sin\left(\frac{m-r}{n}\pi\right)\alpha_j^{-r}=\frac{n}{2\sqrt{-1}}e^{\frac{m\pi}{n}\sqrt{-1}}.$$

In the same way when $j=n$, we find
$$\Sigma_r\sin\left(\frac{m-r}{n}\pi\right)\alpha_j^{-r}=-\frac{n}{2\sqrt{-1}}e^{\frac{-m\pi}{n}\sqrt{-1}}.$$

It results therefore that, according as j is less than $n-1$, equal to $n-1$, or equal to n, we shall have
$$C_j=0,\ \text{ or }\ \frac{m}{n\pi}\frac{e^{\frac{m}{n}\pi\sqrt{-1}}}{2\sqrt{-1}},\ \text{ or }\ \frac{-m}{n\pi}\frac{e^{\frac{-m}{n}\pi\sqrt{-1}}}{2\sqrt{-1}}.$$

In the general integral (II), art. 7, we shall therefore have
$$\Sigma C_i=\frac{m}{n\pi}\left(\frac{e^{\frac{m\pi}{n}\sqrt{-1}}-e^{-\frac{m\pi}{n}\sqrt{-1}}}{2\sqrt{-1}}\right)=\frac{m}{n\pi}\sin\frac{m\pi}{n},$$

$$u=1+\frac{m}{n\pi}\left\{\frac{e^{\frac{m\pi}{n}\sqrt{-1}}}{2\sqrt{-1}}\int_0^\infty dv\,v^{\frac{m}{n}-1}\log(1-\alpha_{n-1}x\mathrm{V})-\frac{e^{\frac{-m\pi}{n}\sqrt{-1}}}{2\sqrt{-1}}\int_0^\infty dv\,v^{\frac{m}{n}-1}\log(1-\alpha_n x\mathrm{V})\right\},\quad(5)$$

where $\mathrm{V}=\frac{v^{\frac{n-1}{n}}}{1+v}$.

Now $$\alpha_{n-1}=e^{(2(n-1)+1)\pi\sqrt{-1}}=e^{(2n-1)\pi\sqrt{-1}}=e^{-\pi\sqrt{-1}},$$

$$\alpha_n\ =e^{(2n+1)\pi\sqrt{-1}}\quad =e^{\pi\sqrt{-1}};$$

therefore, finally,

$$u=1+\frac{m}{2n\pi\sqrt{-1}}\left\{e^{\frac{m\pi}{n}\sqrt{-1}}\int_0^\infty dv\ v^{\frac{m}{n}-1}\log\left(1-e^{\frac{-\pi}{n}\sqrt{-1}}x\mathrm{V}\right)-e^{\frac{-m\pi}{n}\sqrt{-1}}\int_0^\infty dv\ v^{\frac{m}{n}-1}\log\left(1-e^{\frac{\pi}{n}\sqrt{-1}}x\mathrm{V}\right)\right\}. \quad (6)$$

It is seen, from the form of this expression, that it represents a *real* value.

If we substitute v for $v^{\frac{1}{n}}$, a change which does not affect the limits, there results

$$u=y^m=1+\frac{m}{2\pi\sqrt{-1}}\left\{e^{\frac{m\pi}{n}\sqrt{-1}}\int_0^\infty dv\ v^{m-1}\log\left(1-e^{\frac{-\pi}{n}\sqrt{-1}}x\mathrm{V}\right)-e^{-\frac{m\pi}{n}\sqrt{-1}}\int_0^\infty dv\ v^{m-1}\log\left(1-e^{\frac{\pi}{n}\sqrt{-1}}x\mathrm{V}\right)\right\}, \quad \text{(III)}$$

in which $\mathrm{V}=\frac{v^{n-1}}{1+v^n}$. This expression we shall now reduce to an equivalent *real* form.

Reduction of the expression for y^m.

9. We shall somewhat simplify the general expression above found for y^m by integrating by parts. The integrated portion will be found to vanish at both limits.

Representing $\frac{d\mathrm{V}}{dv}$ by V', we have

$$\int mv^{m-1}\log\left(1-e^{\frac{\pm\pi}{n}\sqrt{-1}}x\mathrm{V}\right)dv=v^m\log\left(1-e^{\frac{\pm\pi}{n}\sqrt{-1}}x\mathrm{V}\right)+xe^{\frac{\pm\pi}{n}\sqrt{-1}}\int\frac{v^m\mathrm{V}'dv}{1-e^{\frac{\pm\pi}{n}\sqrt{-1}}x\mathrm{V}}.$$

Now, expanding the logarithm in the integrated portion, and putting for V its value $\frac{v^{n-1}}{1+v^n}$, we see that that portion will consist of a series of terms of the form

$$\frac{\mathrm{A}v^{m+(n-1)r}}{(1+v^n)^r},$$

r being for each such term a positive integer.

All these terms vanish when $v=0$, since, by the conditions to which m is subject, $m+(n-1)r$ is positive.

Again, they vanish when v is made infinite, since in this case

$$\frac{\mathrm{A}v^{m+(n-1)r}}{(1+v^n)^r}=\mathrm{A}v^{m-r},$$

and, by the conditions relative to m, the index $m-r$ is negative.

We have, then, on applying the above reduction to the terms of the general value of y^m,

$$y^m=1+\frac{1}{2\pi\sqrt{-1}}\left\{e^{\frac{(m-1)\pi}{n}\sqrt{-1}}\int_0^\infty\frac{xv^m\mathrm{V}'dv}{1-x\mathrm{V}e^{-\frac{\pi}{n}\sqrt{-1}}}-e^{\frac{-(m-1)\pi}{n}\sqrt{-1}}\int_0^\infty\frac{xv^m\mathrm{V}'dv}{1-x\mathrm{V}e^{\frac{\pi}{n}\sqrt{-1}}}\right\}.$$

Now substitute for the imaginary exponentials their trigonometrical value, and there results

$$y^m=1+\frac{x}{\pi}\int_0^\infty \frac{\left(\text{s n}\left(\frac{m-1}{n}\pi\right)-x\text{V}\sin\frac{m\pi}{n}\right)v^m\text{V}'dv}{1-2x\text{V}\cos\frac{\pi}{n}+x^2\text{V}^2}.$$

As x enters this expression only in combination with V, it is suggested to us to represent xV by V. If we do this the final theorem will be

THEOREM. *If* y^m *represent the* m*th power of that real root of the equation*

$$y^n-xy^{n-1}-1=0$$

which reduces to 1 *when* x=0, *then, supposing* m *to be between the limits* 1 *and* −n+1, *the value of* y^m *will be*

$$y^m=1+\frac{1}{\pi}\int_0^\infty \frac{\left(\sin\left(\frac{m-1}{n}\pi\right)-\text{V}\sin\frac{m\pi}{n}\right)v^m\frac{d\text{V}}{dv}dv}{1-2\text{V}\cos\frac{\pi}{n}+\text{V}^2}, \quad \ldots\ldots\ldots \quad \text{(IV)}$$

in which

$$\text{V}=\frac{xv^{n-1}}{1+v^n}.$$

10. Hence too we have the value of a remarkable definite integral, viz.

$$\int_0^\infty \frac{\left(\sin\frac{m-1}{n}\pi-\text{V}\sin\frac{m\pi}{n}\right)\frac{d\text{V}}{dv}v^m dv}{1-2\text{V}\cos\frac{\pi}{n}+\text{V}^2}=\pi(y^m-1) \quad \ldots\ldots \quad \text{(V)}$$

under the above conditions and with the above interpretations.

It may be desirable to verify this result.

Since
$$\text{V}=\frac{xv^{n-1}}{1+v^n},$$

we shall have
$$\frac{d\text{V}}{dv}=\frac{(n-1)\text{V}}{v}-\frac{n\text{V}^2}{x},$$

so that the definite integral is resolvable into

$$(n-1)\int_0^\infty \frac{v^{m-1}\text{V}\left(\sin\frac{(m-1)\pi}{n}-\text{V}\sin\frac{m\pi}{n}\right)\text{V}'dv}{1-2\text{V}\cos\frac{\pi}{n}+\text{V}^2}$$

$$\frac{-n}{x}\int_0^\infty \frac{v^m\text{V}^2\left(\sin\frac{(m-1)\pi}{n}-\text{V}\sin\frac{m\pi}{n}\right)dv}{1-2\text{V}\cos\frac{\pi}{n}+\text{V}^2}.$$

Now it may be shown that

$$\frac{\sin\left(\frac{m-1}{n}\pi\right)-\mathrm{V}\sin\frac{m\pi}{n}}{1-2\mathrm{V}\cos\frac{\pi}{n}+\mathrm{V}^2}=\Sigma_r\sin\left(\frac{m-r-1}{n}\pi\right)\mathrm{V}^r,$$

the summation with respect to r extending from $r=0$ to $r=\infty$. Hence the first member of (V) may be developed in the form

$$(n-1)\Sigma_r\int_0^\infty v^{m-1}\sin\left(\frac{m-r-1}{n}\pi\right)\mathrm{V}^{r+1}dv$$

$$\frac{-n}{x}\Sigma_r\int_0^\infty v^m\sin\left(\frac{m-r-1}{n}\pi\right)\mathrm{V}^{r+2}\,dv.$$

Now
$$\int_0^\infty v^{m-1}\mathrm{V}^{r+1}dv=x^{r+1}\int_0^\infty\frac{v^{m+(r+1)(n-1)}dv}{(1+v^n)^{r+1}}$$

$$=\frac{\Gamma\left(\frac{m+(r+1)(n-1)}{n}\right)\Gamma\left(\frac{r-m+1}{n}\right)}{n\Gamma(r+1)}x^{r+1},$$

and
$$\int_0^\infty v^m\mathrm{V}^{r+1}dv=x^{r+2}\frac{\Gamma\left(\frac{m+1+(r+2)(n-1)}{n}\right)\Gamma\left(\frac{r-m+1}{n}\right)}{n\Gamma(r+2)}x^{r+2}$$

$$=\frac{m+(r+1)(n-1)}{n(r+1)}\,\frac{\Gamma\left(\frac{m+(r+1)(n-1)}{n}\right)\Gamma\left(\frac{r-m+1}{n}\right)}{n\Gamma(r+1)}x^{r+2}.$$

Hence the total coefficient of x^{r+1} in (V) is

$$\sin\frac{(m-r-1)\pi}{n}\,\frac{\Gamma\left(\frac{m+(r+1)(n-1)}{n}\right)\Gamma\left(\frac{r-m+1}{n}\right)}{n\Gamma(r+1)}\left\{n-1-n\times\frac{m+(r+1)(n-1)}{n(r+1)}\right\}$$

$$=\frac{\sin\frac{(m-r-1)\pi}{n}\Gamma\left(\frac{m+(r+1)(n-1)}{n}\right)\Gamma\left(\frac{r-m+1}{n}\right)}{n\Gamma(r+1)}\times\frac{-m}{r+1},$$

and therefore that of x^r is

$$\frac{\sin\left(\frac{m-r}{n}\pi\right)\Gamma\left(\frac{m+r(n-1)}{n}\right)\Gamma\left(\frac{r-m}{n}\right)}{n\Gamma(r)}\times\frac{-m}{r}.$$

Now

$$\Gamma\left(\frac{m+r(n-1)}{n}\right)\Gamma\left(\frac{r-m}{n}\right)=\Gamma\left(\frac{r-m}{n}\right)\Gamma\left(r-\frac{r-m}{n}\right)$$

$$=\left[r-1-\frac{r-m}{n}\right]^{r-1}\frac{\pi}{\sin\left(\frac{r-m}{n}\pi\right)}.$$

Therefore the coefficient of x^r is

$$\frac{m\left[r-1-\frac{r-m}{n}\right]^{r-1}\pi}{n[r]^r};$$

and this is, by art. 2, equal to πu_r in the expansion of y^m in ascending powers of x. Hence, the lowest value of r in the expansion of the definite integral being unity, we see that the value of that integral will be expressed by $\pi(y^m-1)$, as was to be shown.

It will be observed that the function under the sign of definite integration does not become infinite within the limits. Ordinary methods of approximation might therefore be applied. I apprehend, however, that it is not in this direction that the value of such results is to be sought.